

# The $gl(1|1)$ Lie superbialgebras

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## Abstract

By direct calculations of matrix form of super Jacobi and mixed super Jacobi identities which are obtained from adjoint representation, and using the automorphism supergroup of the  $gl(1|1)$  Lie superalgebra, we determine and classify all  $gl(1|1)$  Lie superbialgebras. Then, by calculating their classical r-matrices, the  $gl(1|1)$  coboundary Lie superbialgebras and their types (triangular, quasi-triangular or factorizable) are determined, furthermore in this way super Poisson structures on the  $\mathbf{GL}(1|1)$  Lie supergroup are obtained. Also, we classify Drinfeld superdoubles based on the  $gl(1|1)$  as a theorem. Finally, as a physical application of the coboundary Lie superbialgebras, we construct a new integrable system on the homogeneous superspace  $OSp(1|2)/U(1)$ .

**keywords:** Lie superbialgebra, Classical r-matrix, Drinfeld superdouble, Integrable system.

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# 1 Introduction

From the mathematical point of view, the classification of Lie bialgebras can be seen as the first step in the classification of quantum groups. Many interesting examples of Lie bialgebras based on complex semisimple Lie algebras have been given by Drinfeld [1]. A complete classification of Lie bialgebras with reduction was given in [2]. However, a classification of Lie bialgebras is out of reach, with similar reasons as for Lie algebra classification. In the non-semisimple case, only a bunch of low dimensional examples have been thoroughly studied [3, 4, 5]. On the other hand, from the physical point of view, the theory of classical integrable systems naturally relates to the geometry and representation theory of Poisson-Lie groups and the corresponding Lie bialgebras and their classical r-matrices (see for example [6]). In the same way, Lie superbialgebras [7], as the underlying symmetry algebras, play an important role in the integrable structure of *AdS/CFT* correspondence [8]. Similarly, one can consider Poisson-Lie T-dual sigma models on Poisson-Lie supergroups [9]. In this way, and by considering that there is a universal quantization for Lie superbialgebras [10], one can assign an important role to the classification of Lie superbialgebras (especially low dimensional Lie superbialgebras) from both, physical and mathematical point of view. Until now there was distinguished and nonsystematic ways for obtaining low dimensional Lie superbialgebras (see for example [11, 12]). In the previous paper, we presented a systematic way for obtaining and classifying low dimensional Lie superbialgebras using the adjoint representation of Lie superalgebras [13]. Here, we apply this method to the classification of the  $gl(1|1)$  Lie superbialgebras. Note that the classification of the quantum deformations of the  $gl(1|1)$  Lie superalgebra was obtained previously in Ref. [14]. Indeed here, we present the classical version of that work in another method. Furthermore, the physical models on the  $\mathbf{GL}(1|1)$  Lie supergroup such as WZW model and its relation to logarithmic CFT have been recently studied [15] (see, also [16]). So, for these reasons, obtaining of the  $gl(1|1)$  Lie superbialgebras is the first step in the presentation of super Poisson-Lie symmetry [9] in such models.

The paper is organized as follows. In section two, we review some basic definitions and notations that are used throughout the paper. Matrix form of super Jacobi and mixed super Jacobi identities [13] of Lie superbialgebras is rewritten in that section. A list of indecomposable four-dimensional Lie superalgebras of the type  $(2, 2)$  [17] is offered in section three. Furthermore, in that section we give the complete list of decomposable Lie superalgebras of the type  $(2, 2)$ . The automorphism Lie supergroup of the  $gl(1|1)$  Lie superalgebra is also presented in section three. Then, using the method mentioned in [13], we classify all  $gl(1|1)$  Lie superbialgebras in section four; the details of calculations are also explained. In section five, by calculating the classical r-matrices of the  $gl(1|1)$  Lie superbialgebras, we determine their types, furthermore in this way, we obtain all super Poisson structures on the  $\mathbf{GL}(1|1)$  Lie supergroup. Section six is devoted to the classification of the Drinfeld superdoubles based on the  $gl(1|1)$ ; the isomorphism matrices between all the Manin supertriples generated by  $gl(1|1)$  have given in appendix A. Finally, in section seven, by using the classical r-matrices of the  $gl(1|1)$  we construct a new integrable system on a supersymplectic supermanifold, namely the homogeneous superspace  $OSp(1|2)/U(1)$ . Some remarks are given in the conclusion section.

## 2 Definitions and notations

Here, we apply DeWitt notations for supervector spaces, supermatrices, etc [18]. In this way, for self-containing of the paper, we recall some basic definitions and a proposition on Lie superbialgebras [4, 7, 13].

**Definition 1** *A Lie superalgebra  $\mathcal{G}$  is a graded vector space  $\mathcal{G} = \mathcal{G}_B \oplus \mathcal{G}_F$  with gradings  $\text{grade}(\mathcal{G}_B) = 0$  and  $\text{grade}(\mathcal{G}_F) = 1$  so that Lie bracket satisfies the super antisymmetric and super Jacobi identities, i.e., in the graded basis  $\{X_i\}$  of  $\mathcal{G}$  we have*<sup>1</sup>

$$[X_i, X_j] = f^k_{ij} X_k, \quad (1)$$

and

$$(-1)^{i(j+k)} f^m_{jl} f^l_{ki} + f^m_{il} f^l_{jk} + (-1)^{k(i+j)} f^m_{kl} f^l_{ij} = 0, \quad (2)$$

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<sup>1</sup> Note that the bracket of one boson with one boson or one fermion is usual commutator, but for one fermion with one fermion is anticommutator. Furthermore, we identify grading of indices by the same indices in the power of  $(-1)$ , for example  $\text{grading}(i) \equiv i$ ; this is the notation that DeWitt applied in [18]. Meanwhile we work in the standard basis [18], i.e., in writing the basis of Lie superalgebras, we consider bosonic generators before fermionic ones.

so that

$$f^k_{ij} = -(-1)^{ij} f^k_{ji}. \quad (3)$$

Furthermore, we have

$$f^k_{ij} = 0, \quad \text{if} \quad \text{grade}(i) + \text{grade}(j) \neq \text{grade}(k) \pmod{2}. \quad (4)$$

Note that using the adjoint representation

$$(\mathcal{X}_i)_j^k = -f^k_{ij}, \quad (\mathcal{Y}^i)_{jk} = -f^i_{jk}, \quad (5)$$

the super Jacobi identities can be rewritten in the following matrix form [13]:

$$(\mathcal{X}_i)_j^k \mathcal{X}_k - \mathcal{X}_j \mathcal{X}_i + (-1)^{ij} \mathcal{X}_i \mathcal{X}_j = 0. \quad (6)$$

Let  $\mathcal{G}$  be a finite-dimensional Lie superalgebra and  $\mathcal{G}^*$  be its dual superspace with respect to a non-degenerate canonical pairing  $(\cdot, \cdot)$  on  $\mathcal{G}^* \oplus \mathcal{G}$ .

**Definition 2** A Lie superbialgebra structure on a Lie superalgebra  $\mathcal{G}$  is a linear map  $\delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$  (the super cocommutator) so that

1)  $\delta$  is a super one-cocycle, i.e.,

$$\delta([X, Y]) = (ad_X \otimes I + I \otimes ad_X) \delta(Y) - (-1)^{|X||Y|} (ad_Y \otimes I + I \otimes ad_Y) \delta(X), \quad \forall X, Y \in \mathcal{G}, \quad (7)$$

where  $|X|(|Y|)$  indicates the grading of  $X(Y)$ .

2) The dual map  ${}^t\delta : \mathcal{G}^* \otimes \mathcal{G}^* \rightarrow \mathcal{G}^*$  is a Lie superbracket on  $\mathcal{G}^*$ , i.e.,

$$(\xi \otimes \eta, \delta(X)) = ({}^t\delta(\xi \otimes \eta), X) = ([\xi, \eta]_*, X), \quad \forall X \in \mathcal{G}; \quad \xi, \eta \in \mathcal{G}^*. \quad (8)$$

The Lie superbialgebra defined in this way will be denoted by  $(\mathcal{G}, \mathcal{G}^*)$  or  $(\mathcal{G}, \delta)$  [7, 13].

**Definition 3** A Lie superbialgebra is coboundary if the super cocommutator is a one-coboundary, i.e., if there exists an element  $r \in \mathcal{G} \otimes \mathcal{G}$  such that

$$\delta(X) = (I \otimes ad_X + ad_X \otimes I)r, \quad \forall X \in \mathcal{G}. \quad (9)$$

**Proposition 4** Two coboundary Lie superbialgebras  $(\mathcal{G}, \mathcal{G}^*)$  and  $(\mathcal{G}', \mathcal{G}'^*)$  defined by  $r \in \mathcal{G} \otimes \mathcal{G}$  and  $r' \in \mathcal{G}' \otimes \mathcal{G}'$  are isomorphic if and only if there is an isomorphism of Lie superalgebras  $\alpha : \mathcal{G} \longrightarrow \mathcal{G}'$  such that  $(\alpha \otimes \alpha)r - r'$  is  $\mathcal{G}'$  invariant [4], i.e.,

$$(I \otimes ad_X + ad_X \otimes I)((\alpha \otimes \alpha)r - r') = 0, \quad \forall X \in \mathcal{G}'. \quad (10)$$

**Definition 5** Coboundary Lie superbialgebras can be of two different types:

(a) if we denote  $r = r^{ij} X_i \otimes X_j$  and  $r$  be a super skew-symmetric solution ( $r^{ij} = -(-1)^{ij} r^{ji}$ ) of the graded classical Yang-Baxter equation (GCYBE)

$$[[r, r]] = 0, \quad (11)$$

then the coboundary Lie superbialgebra is said to be triangular; where the Schouten superbracket is defined by

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \quad (12)$$

such that  $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$ ,  $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$  and  $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$ . A solution of the GCYBE is often called a classical  $r$ -matrix. With regard to the fact  $r$  has even Grassmann parity and Grassmann parity of  $r^{ij}$  comes from indices, i.e.,  $r^{ij} = 0$  if  $i + j = 1$ , one can show that

$$\begin{aligned} [r_{12}, r_{13}] &= (-1)^{i(k+l)+jl} r^{ij} r^{kl} [X_i, X_k] \otimes X_j \otimes X_l, \\ [r_{12}, r_{23}] &= (-1)^{(i+j)(k+l)} r^{ij} r^{kl} X_i \otimes [X_j, X_k] \otimes X_l, \\ [r_{13}, r_{23}] &= (-1)^{i(k+l)+jl} r^{ij} r^{kl} X_i \otimes X_k \otimes [X_j, X_l], \end{aligned}$$

(b) if  $r$  is a solution of GCYBE such that  $r_{12} + r_{21}$  is a  $\mathcal{G}$  invariant element of  $\mathcal{G} \otimes \mathcal{G}$ , then the coboundary Lie superbialgebra is said to be quasi-triangular. Moreover, if the super symmetric part of  $r$  is invertible, then  $r$  is called factorizable.

Sometimes condition (b) can be replaced with the following:

(b') if  $r$  is a super skew-symmetric solution of the modified GCYBE

$$[[r, r]] = \varpi, \quad \varpi \in \wedge^3 \mathcal{G}, \quad (13)$$

then the coboundary Lie superbialgebra is said to be quasi-triangular [7].

Note that if  $\mathcal{G}$  is a Lie superbialgebra then  $\mathcal{G}^*$  is also a Lie superbialgebra [7], but this is not always true for the coboundary property.

**Definition 6** A Manin supertriple is a triple of Lie superalgebras  $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$  together with a nondegenerate ad-invariant supersymmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D}$ , such that

- (1)  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  are Lie subsuperalgebras of  $\mathcal{D}$ ,
- (2)  $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$  as a supervector space,
- (3)  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  are isotropic with respect to  $\langle \cdot, \cdot \rangle$ , i.e.,

$$\langle X_i, X_j \rangle = \langle \tilde{X}^i, \tilde{X}^j \rangle = 0, \quad \delta_i^j = \langle X_i, \tilde{X}^j \rangle = (-1)^{ij} \langle \tilde{X}^j, X_i \rangle = (-1)^{ij} \delta_i^j, \quad (14)$$

where  $\{X_i\}$  and  $\{\tilde{X}^i\}$  are basis of Lie superalgebras  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ , respectively [7, 13].

Note that in the second equation of the (14),  $\delta_i^j$  is the ordinary delta function. There is a one-to-one correspondence between Lie superbialgebra  $(\mathcal{G}, \mathcal{G}^*)$  and Manin supertriple  $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$  with  $\tilde{\mathcal{G}} = \mathcal{G}^*$ . If we choose the structure constants of Lie superalgebras  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  as

$$[X_i, X_j] = f_{ij}^k X_k, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}_k \tilde{X}^k, \quad (15)$$

then ad-invariance of the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$  implies that [13]

$$[X_i, \tilde{X}^j] = (-1)^j \tilde{f}^{jk}_i X_k + (-1)^i f^j_{ki} \tilde{X}^k. \quad (16)$$

Clearly, using Eqs. (8), (14) and (16) we have

$$\delta(X_i) = (-1)^{jk} \tilde{f}^{jk}_i X_j \otimes X_k, \quad (17)$$

note that the appearance of  $(-1)^{jk}$  in this relation is due to the definition of natural inner product between  $\mathcal{G} \otimes \mathcal{G}$  and  $\mathcal{G}^* \otimes \mathcal{G}^*$  as  $(\tilde{X}^i \otimes \tilde{X}^j, X_k \otimes X_l) = (-1)^{jk} \delta_k^i \delta_l^j$ .

As a result, if we apply this relation in the super one-cocycle condition (7), the super Jacobi identities (2) for the dual Lie superalgebra and the following mixed super Jacobi identities are obtained:

$$f_{jk}^m \tilde{f}^{il}_m = f_{mk}^i \tilde{f}^{ml}_j + f_{jm}^l \tilde{f}^{im}_k + (-1)^{jl} f_{jm}^i \tilde{f}^{ml}_k + (-1)^{ik} f_{mk}^l \tilde{f}^{im}_j. \quad (18)$$

This relation can also be obtained from super Jacobi identity of  $\mathcal{D}$ . As mentioned in Ref. [13], using the adjoint representation, the matrix form of mixed super Jacobi identities has the following form:

$$(\tilde{\mathcal{X}}^i)^j_l \mathcal{Y}^l = -(-1)^k (\tilde{\mathcal{X}}^{st})^j \mathcal{Y}^i + \mathcal{Y}^j \tilde{\mathcal{X}}^i - (-1)^{ij} \mathcal{Y}^i \tilde{\mathcal{X}}^j + (-1)^{k+ij} (\tilde{\mathcal{X}}^{st})^i \mathcal{Y}^j, \quad (19)$$

where index  $k$  represents the column of matrix  $\tilde{\mathcal{X}}^{st}$  and superscript  $st$  stands for supertranspose.

### 3 Lie superalgebras of the type (2, 2)

In this section, we use the classification of four-dimensional Lie superalgebras of the type (2, 2) listed in [17]. In that classification, Lie superalgebras are divided into two types: trivial and nontrivial Lie superalgebras for which the commutation of fermion-fermion is zero or nonzero, respectively [As we use DeWitt notation here, the structure constant  $C_{FF}^B$  must be pure imaginary in the standard basis.]. The results have been presented in tables I and II. Note that in [17] only the indecomposable Lie superalgebras are classified. Here we also consider decomposable Lie superalgebras of the type (2, 2) as presented in table III. As the tables indicate,

the Lie superalgebras have two,  $\{X_1, X_2\}$  bosonic and two,  $\{X_3, X_4\}$  fermionic generators. The labeling of the trivial Lie superalgebras is such that the letters A, B, C and D with integral superscripts  $i$  and real subscripts  $p$  and  $q$ , denote the equivalence classes of Lie superalgebras of dimension  $d$ , where  $d=1$  for A,  $d=2$  for B,  $d=3$  for C and  $d=4$  for D. The superscript  $i$  and real subscripts  $p$  and  $q$  denote the number of non isomorphic Lie superalgebras and the Lie superalgebras parameters, respectively. For the nontrivial Lie superalgebras, we add the bracketed symbol to the corresponding trivial Lie superalgebra, where necessary, an integral superscript and a real subscript.

TABLE I. Trivial indecomposable Lie superalgebras of the type  $(2, 2)$ .

$\mathcal{G}$	Non-zero commutation relations	Comments
$D^5$	$[X_1, X_3] = X_3, [X_1, X_4] = X_4, [X_2, X_4] = X_3$	
$D^6$	$[X_1, X_3] = X_3, [X_1, X_4] = X_4, [X_2, X_3] = -X_4, [X_2, X_4] = X_3$	
$D_{pq}^1$	$[X_1, X_2] = X_2, [X_1, X_3] = pX_3, [X_1, X_4] = qX_4$	$pq \neq 0, p \geq q$
$D_p^8$	$[X_1, X_2] = X_2, [X_1, X_3] = pX_3, [X_1, X_4] = X_3 + pX_4$	$p \neq 0$
$D_{pq}^9$	$[X_1, X_2] = X_2, [X_1, X_3] = pX_3 - qX_4, [X_1, X_4] = qX_3 + pX_4$	$q > 0$
$D_p^{10}$	$[X_1, X_2] = X_2, [X_1, X_3] = (p+1)X_3, [X_1, X_4] = pX_4, [X_2, X_4] = X_3$	

TABLE II. Non-trivial indecomposable Lie superalgebras of the type  $(2, 2)$ .

$\mathcal{G}$	Non-zero (anti) commutation relations	Comments
$(D_{\frac{1}{2}, \frac{1}{2}}^7)^1$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3, [X_1, X_4] = \frac{1}{2}X_4, \{X_3, X_3\} = iX_2, \{X_4, X_4\} = iX_2$	
$(D_{\frac{1}{2}, \frac{1}{2}}^7)^2$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3, [X_1, X_4] = \frac{1}{2}X_4, \{X_3, X_3\} = iX_2, \{X_4, X_4\} = -iX_2$	
$(D_{\frac{1}{2}, \frac{1}{2}}^7)^3$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3, [X_1, X_4] = \frac{1}{2}X_4, \{X_3, X_3\} = iX_2$	
$(D_{1-p, p}^7)$	$[X_1, X_2] = X_2, [X_1, X_3] = pX_3, [X_1, X_4] = (1-p)X_4, \{X_3, X_4\} = iX_2$	$p \leq \frac{1}{2}$
$(D_{\frac{1}{2}}^8)$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3, [X_1, X_4] = X_3 + \frac{1}{2}X_4, \{X_4, X_4\} = iX_2$	
$(D_{\frac{1}{2}, p}^9)$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3 - pX_4, [X_1, X_4] = pX_3 + \frac{1}{2}X_4, \{X_3, X_3\} = iX_2, \{X_4, X_4\} = iX_2$	$p > 0$
$(D_0^{10})^1$	$[X_1, X_2] = X_2, [X_1, X_3] = X_3, [X_2, X_4] = X_3, \{X_4, X_4\} = iX_1, \{X_3, X_4\} = -i\frac{1}{2}X_2$	
$(D_0^{10})^2$	$[X_1, X_2] = X_2, [X_1, X_3] = X_3, [X_2, X_4] = X_3, \{X_4, X_4\} = -iX_1, \{X_3, X_4\} = i\frac{1}{2}X_2$	
$(2A_{1,1} + 2A)^2$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = iX_2, \{X_3, X_4\} = iX_1$	Nilpotent
$(2A_{1,1} + 2A)_p^3$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = iX_2, \{X_3, X_4\} = ip(X_1 + X_2)$	$p > 0$ Nilpotent
$(2A_{1,1} + 2A)_p^4$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = iX_2, \{X_3, X_4\} = ip(X_1 - X_2)$	$p > 0$ Nilpotent
$(C_1^1 + A)$	$[X_1, X_2] = X_2, [X_1, X_3] = X_3, \{X_3, X_4\} = iX_2$	
$(C_{-1}^2 + A)$	$[X_1, X_3] = X_3, [X_1, X_4] = -X_4, \{X_3, X_4\} = iX_2$	Jordan-Wigner quantization
$(C^3 + A)$	$[X_1, X_4] = X_3, \{X_4, X_4\} = iX_2$	Nilpotent
$(C_0^5 + A)$	$[X_1, X_3] = -X_4, [X_1, X_4] = X_3, \{X_3, X_3\} = iX_2, \{X_4, X_4\} = iX_2$	

The Lie superalgebra  $A$  is one dimensional Abelian Lie superalgebra with one fermionic generator where Lie superalgebra  $A_{1,1}$  is its bosonization.

Note that, as mentioned in Ref. [19] the four-dimensional Drinfeld superdoubles are equal or isomorphic to some of the Lie superalgebras of table II in the following way

$$(C^3 + A) \equiv ((A_{1,1} + A), I_{(1,1)}),$$

$$(C_{-1}^2 + A) \cong (B, I_{(1,1)}) \cong (B, (A_{1,1} + A)) \cong (B, (A_{1,1} + A).i).$$

So four-dimensional Drinfeld superdoubles have no new results for table II. As mentioned above, in the following table we give all possible decomposable Lie superalgebras of the type  $(2, 2)$ .

TABLE III. Decomposable Lie superalgebras of the type  $(2, 2)$ .

	$\mathcal{G}$	Non-zero (anti)commutation relations	Comments
Trivial	$I_{(2,2)}$	All of the (anti)commutation relations are zero.	
	$B \oplus B$	$[X_1, X_3] = X_3, [X_2, X_4] = X_4$	
	$C_p^1 \oplus A$	$[X_1, X_2] = X_2, [X_1, X_3] = pX_3$	$p \in \mathfrak{R} - \{0\}$
	$C_p^2 \oplus A_{1,1}$	$[X_1, X_3] = X_3, [X_1, X_4] = pX_4$	$0 <  p  \leq 1$
	$L \oplus 2A$	$[X_1, X_2] = X_2$	$\equiv C_{p=0}^1 \oplus A$
	$B \oplus A \oplus A_{1,1}$	$[X_1, X_3] = X_3$	$\equiv C_{p=0}^2 \oplus A_{1,1}$
	$C^3 \oplus A_{1,1}$	$[X_1, X_4] = X_3$	Nilpotent
	$C^4 \oplus A_{1,1}$	$[X_1, X_3] = X_3, [X_1, X_4] = X_3 + X_4$	
	$C_p^5 \oplus A_{1,1}$	$[X_1, X_3] = pX_3 - X_4, [X_1, X_4] = X_3 + pX_4$	$p \geq 0$
Nontrivial	$(2A_{1,1} + 2A)^0$	$\{X_3, X_3\} = iX_1$	$\equiv (A_{1,1} + A) \oplus A \oplus A_{1,1}$ , Nilpotent
	$(2A_{1,1} + 2A)^1$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = iX_2$	$\equiv (A_{1,1} + A) \oplus (A_{1,1} + A)$ , Nilpotent
	$(B \oplus (A_{1,1} + A))$	$[X_1, X_3] = X_3, \{X_4, X_4\} = iX_2$	
	$((A_{1,1} + 2A)^1 \oplus A_{1,1})$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = iX_1$	Nilpotent
	$((A_{1,1} + 2A)^2 \oplus A_{1,1})$	$\{X_3, X_3\} = iX_1, \{X_4, X_4\} = -iX_1$	Nilpotent
	$(C_{\frac{1}{2}}^1 + A)$	$[X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2}X_3, \{X_3, X_3\} = iX_2$	$\equiv (C_{\frac{1}{2}}^1) \oplus A$

$L$  is a two-dimensional bosonic Lie algebra and  $(A_{1,1} + A)$  is regarded as a subsuperalgebra of  $sl(1|1) \subset gl(1|1)$ .

Note that the Lie superalgebras of  $(2A_{1,1} + 2A)^1$  and  $(C_{\frac{1}{2}}^1 + A)$  are decomposable Lie superalgebras such that in Backhouse's classification [17] have been given in the list of indecomposable Lie superalgebras.

### 3.1 The $gl(1|1)$ Lie superalgebra and its automorphism supergroup

Traditional notation for the  $(C_{-1}^2 + A)$  Lie superalgebra is the  $gl(1|1)$ . The  $gl(1|1)$  Lie superalgebra is spanned by the set of generators  $\{X_1, X_2, X_3, X_4\}$  with grading;  $grade(X_1) = grade(X_2) = 0$  and  $grade(X_3) = grade(X_4) = 1$ , which in the standard basis, fulfill the following (anti) commutation relations

$$[X_1, X_3] = X_3, [X_1, X_4] = -X_4, \{X_3, X_4\} = X_2. \quad (20)$$

To obtain the dual Lie superalgebras for the  $gl(1|1)$ , we need automorphism supergroup of this Lie superalgebra. In order to calculate the automorphism supergroup of the  $gl(1|1)$  Lie superalgebra, we use the following transformation [13]:

$$X'_i = (-1)^j A_i^j X_j, \quad [X'_i, X'_j] = f_{ij}^k X'_k, \quad (21)$$

thus, we have the following matrix equation for the elements of automorphism supergroup [13]

$$(-1)^{ij+mk} A \mathcal{Y}^k A^{st} = \mathcal{Y}^e A_e^k, \quad (22)$$

where the indices  $i$  and  $j$  correspond to the row and column of matrix  $\mathcal{Y}^k$ , respectively, and  $m$  denotes the column of matrix  $A^{st}$  in the left hand side. As the Lie superalgebra is generated by just two bosons  $X_1$  and  $X_2$ , every automorphism is determined by the following transformation of the bosons:

$$X'_1 = X_1 + aX_2, \quad X'_2 = bcX_2. \quad (23)$$

For two fermions  $X_3$  and  $X_4$ , we find that under the above transformation

$$X'_3 = -bX_3, \quad X'_4 = -cX_4, \quad (24)$$

where  $a \in R$  and  $b, c \in R - \{0\}$ . Finally, using the Eq. (22) one can derive the matrix form of automorphism supergroup of the  $gl(1|1)$  Lie superalgebra as follows:

$$A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & bc & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}. \quad (25)$$

## 4 The $gl(1|1)$ Lie superbialgebras

To obtain the  $gl(1|1)$  Lie superbialgebras, we must first solve the Eq. (6) for dual Lie superalgebras and Eq. (19), then, by using automorphism supergroup of the  $gl(1|1)$  Lie superalgebra and the method mentioned in [13] we classify the  $gl(1|1)$  Lie superbialgebras. The solutions of super Jacobi (the Eq. (6) for dual Lie superalgebras) and mixed super Jacobi (19) identities for dual Lie superalgebras of the  $gl(1|1)$  have the following four forms:

- *Case A :*

$$\tilde{f}_{11}^{33} = \tilde{f}_{11}^{33}, \quad \tilde{f}_{11}^{44} = \tilde{f}_{11}^{44}, \quad \tilde{f}_{41}^{23} = \frac{-i}{2} \tilde{f}_{11}^{33}, \quad \tilde{f}_{31}^{24} = \frac{i}{2} \tilde{f}_{11}^{44}.$$

In this case, the general form of the (anti) commutation relations for  $\tilde{\mathcal{G}}$  are as follows:

$$[\tilde{X}^2, \tilde{X}^3] = \frac{\alpha}{2} \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = \frac{-\beta}{2} \tilde{X}^3, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\alpha \tilde{X}^1, \quad \{\tilde{X}^4, \tilde{X}^4\} = i\beta \tilde{X}^1, \quad \forall \alpha, \beta \in \mathfrak{R}.$$

- *Case B :*

$$\tilde{f}_3^{23} = \tilde{f}_3^{23}, \quad \tilde{f}_4^{24} = \tilde{f}_4^{24},$$

where the general form of the (anti) commutation relations for  $\tilde{\mathcal{G}}$  are given by

$$[\tilde{X}^2, \tilde{X}^3] = \alpha \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \beta \tilde{X}^4, \quad \forall \alpha, \beta \in \mathfrak{R}.$$

- *Case C :*

$$\tilde{f}_{11}^{44} = \tilde{f}_{11}^{44}, \quad \tilde{f}_{31}^{24} = \frac{i}{2} \tilde{f}_{11}^{44}, \quad \tilde{f}_3^{23} = \tilde{f}_3^{23}.$$

In this case, the general form of the (anti) commutation relations for  $\tilde{\mathcal{G}}$  are as follows:

$$[\tilde{X}^2, \tilde{X}^3] = \alpha \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \frac{-\beta}{2} \tilde{X}^3, \quad \{\tilde{X}^4, \tilde{X}^4\} = i\beta \tilde{X}^1, \quad \forall \alpha, \beta \in \mathfrak{R}.$$

- *Case D :*

$$\tilde{f}_{11}^{33} = \tilde{f}_{11}^{33}, \quad \tilde{f}_{41}^{23} = \frac{-i}{2} \tilde{f}_{11}^{33}, \quad \tilde{f}_4^{24} = \tilde{f}_4^{24},$$

where the general form of the (anti) commutation relations for  $\tilde{\mathcal{G}}$  are given by

$$[\tilde{X}^2, \tilde{X}^4] = \beta \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \frac{\alpha}{2} \tilde{X}^4, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\alpha \tilde{X}^1, \quad \forall \alpha, \beta \in \mathfrak{R}.$$

In the following, we find isomorphism matrices  $C$  between the above solutions and Lie superalgebras of tables I, II and III. In this way, we see that the solutions of super Jacobi and mixed super Jacobi identities for dual Lie superalgebras of the  $gl(1|1)$  are isomorphic to the  $(C^3 + A)$ ,  $(C_{-1}^2 + A)$  and  $(C_0^5 + A)$  Lie superalgebras of table II and the  $B \oplus A \oplus A_{1,1}, C_p^2 \oplus A_{1,1}$  and  $(B \oplus (A_{1,1} + A))$  Lie superalgebras of table III.

- The solution of *Case A* is isomorphic to the  $(C^3 + A)$ ,  $(C_{-1}^2 + A)$  and  $(C_0^5 + A)$  Lie superalgebras such that isomorphism matrices  $C$  between this solution and the  $(C^3 + A)$  Lie superalgebra are as follows:

$$C_1 = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 \\ -i\tilde{f}_1^{44}c_{44}^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{i\tilde{f}_1^{44}}{2}c_{12}c_{44} & 0 \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}, \quad c_{12}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11}, c_{43} \in \mathfrak{R},$$

where  $\tilde{f}_1^{33} = \tilde{f}_4^{23} = 0$ , and

$$C_2 = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 \\ -i\tilde{f}_1^{33}c_{43}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i\tilde{f}_1^{33}}{2}c_{12}c_{43} \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}, \quad c_{12}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11}, c_{43} \in \mathfrak{R},$$

where  $\tilde{f}_1^{44} = \tilde{f}_3^{24} = 0$ .

Furthermore, the isomorphism matrix related to the  $(C_{-1}^2 + A)$  Lie superalgebra is given by

$$C_3 = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 \\ -2i\tilde{f}_1^{44}c_{34}c_{44} & 0 & 0 & 0 \\ 0 & 0 & \frac{i\tilde{f}_1^{44}}{2}c_{12}c_{34} & c_{34} \\ 0 & 0 & -\frac{i\tilde{f}_1^{44}}{2}c_{12}c_{44} & c_{44} \end{pmatrix}, \quad c_{12}, c_{34}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_1^{33} = (\frac{4}{c_{12}^2})\frac{1}{\tilde{f}_1^{44}}$ ; and finally the isomorphism matrix  $C$  between this case and the  $(C_0^5 + A)$  Lie superalgebra is as follows:

$$C_4 = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 \\ -i\tilde{f}_1^{44}(c_{34}^2 + c_{44}^2) & 0 & 0 & 0 \\ 0 & 0 & \frac{i\tilde{f}_1^{44}}{2}c_{12}c_{44} & c_{34} \\ 0 & 0 & -\frac{i\tilde{f}_1^{44}}{2}c_{12}c_{34} & c_{44} \end{pmatrix}, \quad c_{12}, c_{34}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_1^{33} = -(\frac{4}{c_{12}^2})\frac{1}{\tilde{f}_1^{44}}$ .

- The solution of *Case B* is isomorphic to the  $B \oplus A \oplus A_{1,1}$  and  $C_p^2 \oplus A_{1,1}$  Lie superalgebras such that isomorphism matrices  $C$  between this solution and the  $B \oplus A \oplus A_{1,1}$  Lie superalgebra are as follows:

$$C_1 = \begin{pmatrix} c_{11} & \frac{1}{\tilde{f}_3^{23}} & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}, \quad c_{21}, c_{33}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_4^{24} = 0$ , and

$$C_2 = \begin{pmatrix} c_{11} & \frac{1}{\tilde{f}_4^{24}} & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{34} \\ 0 & 0 & c_{43} & 0 \end{pmatrix}, \quad c_{21}, c_{34}, c_{43} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_3^{23} = 0$ .

The isomorphism matrices related to the  $C_p^2 \oplus A_{1,1}$  Lie superalgebras are given by

$$C_3 = \begin{pmatrix} c_{11} & \frac{1}{\tilde{f}_4^{24}} & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}, \quad c_{33}c_{44} - c_{34}c_{43} \neq 0; \quad c_{21} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$



where  $p = 1$  and  $\tilde{f}_3^{23} = \tilde{f}_4^{24}$ ,

$$C_4 = \begin{pmatrix} c_{11} & \frac{p}{\tilde{f}_4^{24}} & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}, \quad c_{21}, c_{33}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $0 < |p| \leq 1$  and  $\tilde{f}_4^{24} = p\tilde{f}_3^{23}$ , and

$$C_5 = \begin{pmatrix} c_{11} & \frac{p}{\tilde{f}_3^{23}} & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{34} \\ 0 & 0 & c_{43} & 0 \end{pmatrix}, \quad c_{21}, c_{34}, c_{43} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $0 < |p| \leq 1$  and  $\tilde{f}_3^{23} = p\tilde{f}_4^{24}$ .

- The solution of *Case C* is isomorphic to the  $(B \oplus (A_{1,1} + A))$ ,  $B \oplus A \oplus A_{1,1}$  and  $(C^3 + A)$  Lie superalgebras such that isomorphism matrix  $C$  between this solution and the  $(B \oplus (A_{1,1} + A))$  Lie superalgebra is as follows:

$$C_1 = \begin{pmatrix} c_{11} & \frac{1}{\tilde{f}_3^{23}} & 0 & 0 \\ 2\tilde{f}_3^{23}c_{43}c_{44} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}, \quad c_{33}, c_{43}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_1^{44} = \frac{2ic_{43}}{c_{44}}\tilde{f}_3^{23}$ . The isomorphism matrix between this solution and the  $B \oplus A \oplus A_{1,1}$  Lie superalgebra is the same as the isomorphism matrix  $C_1$  in the solution of *Case B* with the same conditions. Furthermore, the isomorphism matrix between this solution and the  $(C^3 + A)$  Lie superalgebra is the same as the isomorphism matrix  $C_1$  in the solution of *Case A* with the same conditions.

TABLE IV. Dual Lie superalgebras to  $gl(1|1)$ .

	$\tilde{\mathcal{G}}$	Non-zero (anti) commutation relations	Comments
Trivial	$I_{(2,2)}$	All of the commutation relations are zero.	
	$B \oplus A \oplus A_{1,1} .i$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$	
	$B \oplus A \oplus A_{1,1} .ii$	$[\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$C_{p=1}^2 \oplus A_{1,1} .i$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$C_{p=-1}^2 \oplus A_{1,1} .ii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^4$	
	$C_p^2 \oplus A_{1,1} .i$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = p\tilde{X}^4$	$0 <  p  < 1$
	$C_{\frac{1}{p}}^2 \oplus A_{1,1} .ii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \frac{1}{p}\tilde{X}^4$	$0 <  p  < 1$
Nontrivial	$(B \oplus (A_{1,1} \oplus A))_{\epsilon}.i$	$[\tilde{X}^2, \tilde{X}^4] = \epsilon\tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \frac{\epsilon}{2}\tilde{X}^4, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\epsilon\tilde{X}^1$	
	$(B \oplus (A_{1,1} \oplus A))_{\epsilon}.ii$	$[\tilde{X}^2, \tilde{X}^3] = \epsilon\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = -\frac{\epsilon}{2}\tilde{X}^3, \quad \{\tilde{X}^4, \tilde{X}^4\} = i\epsilon\tilde{X}^1$	
	$(C^3 + A)_{\epsilon}.i$	$[\tilde{X}^2, \tilde{X}^4] = \frac{\epsilon}{2}\tilde{X}^3, \quad \{\tilde{X}^4, \tilde{X}^4\} = -i\epsilon\tilde{X}^1$	Nilpotent
	$(C^3 + A)_{\epsilon}.ii$	$[\tilde{X}^2, \tilde{X}^3] = \frac{\epsilon}{2}\tilde{X}^4, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\epsilon\tilde{X}^1$	Nilpotent
	$(C_{-1}^2 + A).i$	$[\tilde{X}^2, \tilde{X}^3] = \frac{1}{2}\tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = \frac{1}{2}\tilde{X}^3, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\tilde{X}^1, \quad \{\tilde{X}^4, \tilde{X}^4\} = -i\tilde{X}^1$	
	$(C_0^5 + A).i$	$[\tilde{X}^2, \tilde{X}^3] = \frac{1}{2}\tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = -\frac{1}{2}\tilde{X}^3, \quad \{\tilde{X}^3, \tilde{X}^3\} = i\tilde{X}^1, \quad \{\tilde{X}^4, \tilde{X}^4\} = i\tilde{X}^1$	

◦  $\epsilon = \pm 1$ .

- The solution of *Case D* is isomorphic to the  $(B \oplus (A_{1,1} + A))$ ,  $B \oplus A \oplus A_{1,1}$  and  $(C^3 + A)$  Lie superalgebras such that isomorphism matrix  $C$  between this solution and the  $(B \oplus (A_{1,1} + A))$  Lie superalgebra is given

by

$$C_1 = \begin{pmatrix} c_{11} & \frac{1}{\tilde{f}_4^{24}} & 0 & 0 \\ -2\tilde{f}_4^{24}c_{43}c_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}, \quad c_{34}, c_{43}, c_{44} \in \mathfrak{R} - \{0\}; \quad c_{11} \in \mathfrak{R},$$

where  $\tilde{f}_1^{33} = -\frac{2ic_{44}}{c_{43}}\tilde{f}_4^{24}$ . The isomorphism matrix between this solution and the  $B \oplus A \oplus A_{1,1}$  Lie superalgebra is the same as the isomorphism matrix  $C_2$  in the solution of *Case B* with the same conditions. Furthermore, the isomorphism matrix between this solution and the  $(C^3 + A)$  Lie superalgebra is the same as the isomorphism matrix  $C_2$  in the solution of *Case A* with the same conditions.

Now using the automorphism supergroup of the  $gl(1|1)$  and the method mentioned in [13], one can classify all  $gl(1|1)$  Lie superbialgebras; the results are presented in table IV.

## 5 The $gl(1|1)$ coboundary Lie superbialgebras

In this section, we determine which of seventeen  $gl(1|1)$  Lie superbialgebras of table IV are coboundary? Note that here we work in nonstandard basis, so we omit the coefficient  $i = \sqrt{-1}$  from all anticommutation relations for the  $gl(1|1)$  Lie superalgebra and its dual Lie superalgebras of table IV. In this regard, we must find  $r = r^{ij}X_i \otimes X_j \in \mathcal{G} \otimes \mathcal{G}$  such that the super cocommutator of Lie superbialgebras can be written as in the form (9). Using Eqs. (9), (15) and (17), we have [20]

$$\tilde{\mathcal{Y}}_i = \mathcal{X}_i^{st}r + (-1)^l r\mathcal{X}_i, \quad (26)$$

where index  $l$  corresponds to the row of matrix  $\mathcal{X}_i$ . Now using the above relations, we can find r-matrix of Lie superbialgebras and determine which of the presented Lie superbialgebras in table IV are coboundary. We also perform this work for the dual Lie superbialgebras  $(\tilde{\mathcal{G}}, \mathcal{G})$  using the following equations as (26) [20]

$$\mathcal{Y}^i = (\tilde{\mathcal{X}}^i)^{st}\tilde{r} + (-1)^l \tilde{r}\tilde{\mathcal{X}}^i. \quad (27)$$

Among the Lie superbialgebras listed in table IV, only six of them are coboundary and have not coboundary duals. The results are presented in table V.

TABLE V. The  $gl(1|1)$  coboundary Lie superbialgebras.

	$(\mathcal{G}, \tilde{\mathcal{G}})$	$r$	$[[r, r]]$
Triangular	$(gl(1 1), C_{p=-1}^2 \oplus A_{1,1} .ii)$	$X_1 \wedge X_2$	0
Quasi-triangular	$(gl(1 1), B \oplus A \oplus A_{1,1} .i)$	$\frac{1}{2}X_1 \wedge X_2 + \frac{1}{2}X_3 \wedge X_4$	$-\frac{1}{4}X_2 \wedge X_3 \wedge X_4$
	$(gl(1 1), B \oplus A \oplus A_{1,1} .ii)$	$-\frac{1}{2}X_1 \wedge X_2 + \frac{1}{2}X_3 \wedge X_4$	$-\frac{1}{4}X_2 \wedge X_3 \wedge X_4$
	$(gl(1 1), C_{p=1}^2 \oplus A_{1,1} .i)$	$X_3 \wedge X_4$	$-X_2 \wedge X_3 \wedge X_4$
	$(gl(1 1), C_p^2 \oplus A_{1,1} .i)$	$\frac{1-p}{2}X_1 \wedge X_2 + \frac{1+p}{2}X_3 \wedge X_4$	$-\frac{(1+p)^2}{4}X_2 \wedge X_3 \wedge X_4$
	$(gl(1 1), C_{\frac{1}{p}}^2 \oplus A_{1,1} .ii)$	$\frac{p-1}{2p}X_1 \wedge X_2 + \frac{p+1}{2p}X_3 \wedge X_4$	$-\frac{(1+p)^2}{4p^2}X_2 \wedge X_3 \wedge X_4$

### 5.1 Super Poisson structures on the $\mathbf{GL}(1|1)$ Lie supergroup

To obtain the corresponding super Poisson structures on the  $\mathbf{GL}(1|1)$  Lie supergroup, for the coboundary Lie superbialgebras of table V we use the Sklyanin superbracket (the graded Poisson r-bracket) provided by a given super skew-symmetric r-matrix  $r = r^{ij}X_i \otimes X_j$  [7, 11, 20]. For these reasons we need the left and right invariant supervector fields with left (right) derivative on the  $G = \mathbf{GL}(1|1)$  Lie supergroup. To this end, we assume a convenient parametrization of the  $\mathbf{GL}(1|1)$  Lie supergroup by means of exponentiation as follows:

$$g = e^{Z^A X_A} = e^{xX_1} e^{yX_2} e^{\psi X_3} e^{\chi X_4}, \quad g \in G, \quad (28)$$

which results in

$${}_x X^{(L,l)} = \frac{\overrightarrow{\partial}}{\partial x} - \psi \frac{\overrightarrow{\partial}}{\partial \psi} + \chi \frac{\overrightarrow{\partial}}{\partial \chi}, \quad {}_y X^{(L,l)} = \frac{\overrightarrow{\partial}}{\partial y}, \quad {}_\psi X^{(L,l)} = -\chi \frac{\overrightarrow{\partial}}{\partial y} - \frac{\overrightarrow{\partial}}{\partial \psi}, \quad {}_\chi X^{(L,l)} = -\frac{\overrightarrow{\partial}}{\partial \chi}, \quad (29)$$

$$X_x^{(L,r)} = \frac{\overleftarrow{\partial}}{\partial x} - \frac{\overleftarrow{\partial}}{\partial \psi} \psi + \frac{\overleftarrow{\partial}}{\partial \chi} \chi, \quad X_y^{(L,r)} = \frac{\overleftarrow{\partial}}{\partial y}, \quad X_\psi^{(L,r)} = \frac{\overleftarrow{\partial}}{\partial y} \chi - \frac{\overleftarrow{\partial}}{\partial \psi}, \quad X_\chi^{(L,r)} = -\frac{\overleftarrow{\partial}}{\partial \chi}, \quad (30)$$

$${}_x X^{(R,l)} = \frac{\overrightarrow{\partial}}{\partial x}, \quad {}_y X^{(R,l)} = \frac{\overrightarrow{\partial}}{\partial y}, \quad {}_\psi X^{(R,l)} = -e^{-x} \frac{\overrightarrow{\partial}}{\partial \psi}, \quad {}_\chi X^{(R,l)} = \psi e^x \frac{\overrightarrow{\partial}}{\partial y} - e^x \frac{\overrightarrow{\partial}}{\partial \chi}, \quad (31)$$

$$X_x^{(R,r)} = \frac{\overleftarrow{\partial}}{\partial x}, \quad X_y^{(R,r)} = \frac{\overleftarrow{\partial}}{\partial y}, \quad X_\psi^{(R,r)} = -\frac{\overleftarrow{\partial}}{\partial \psi} e^{-x}, \quad X_\chi^{(R,r)} = -\frac{\overleftarrow{\partial}}{\partial y} \psi e^x - \frac{\overleftarrow{\partial}}{\partial \chi} e^x, \quad (32)$$

where  ${}_i X^{(L,l)}(X_i^{(L,r)})$  and  ${}_i X^{(R,l)}(X_i^{(R,r)})$  stand for left invariant supervector fields with left(right) derivative and right invariant supervector fields with left(right) derivative, respectively. Introducing the generators  $Z^A$  representing the supergroup parameters, one can define for the functions  $f(Z^A)$  the graded Poisson r-bracket as follows [20]:

$$\begin{aligned} \{f, h\} &= \{f, h\}^L - \{f, h\}^R \\ &= \frac{f \overleftarrow{\partial}}{\partial x^\mu} {}^\mu X_i^{(L,r)} r^{ij} {}_j X^{(L,l) \nu} \frac{\overrightarrow{\partial} h}{\partial x^\nu} - \frac{f \overleftarrow{\partial}}{\partial x^\mu} {}^\mu X_i^{(R,r)} r^{ij} {}_j X^{(R,l) \nu} \frac{\overrightarrow{\partial} h}{\partial x^\nu}, \quad \forall f, h \in C^\infty(G), \end{aligned} \quad (33)$$

with the following graded anti symmetry property

$$\{f, h\} = -(-1)^{|f||h|} \{h, f\}. \quad (34)$$

Now using the results (29)-(32) and the r-matrices listed in table V, one can calculate the super Poisson structures on the  $\mathbf{GL}(1|1)$  Lie supergroup. The results are listed in the following table:

TABLE VI. Poisson superbrackets the triangular and quasi-triangular the coboundary Lie superbialgebras  $\mathcal{G} = gl(1|1)$ .

$(\mathcal{G}, \mathcal{G})$	$(\mathcal{G}, C_{p=-1}^2 \oplus A_{1,1 } . ii)$				
$\{x, y\}^L$	1	$\{x, y\}^R$	1	$\{x, y\}$	0
$\{x, \psi\}^L$	0	$\{x, \psi\}^R$	0	$\{x, \psi\}$	0
$\{x, \chi\}^L$	0	$\{x, \chi\}^R$	0	$\{x, \chi\}$	0
$\{y, \psi\}^L$	$\psi$	$\{y, \psi\}^R$	0	$\{y, \psi\}$	$\psi$
$\{y, \chi\}^L$	$-\chi$	$\{y, \chi\}^R$	0	$\{y, \chi\}$	$-\chi$
$\{\psi, \psi\}^L$	0	$\{\psi, \psi\}^R$	0	$\{\psi, \psi\}$	0
$\{\psi, \chi\}^L$	0	$\{\psi, \chi\}^R$	0	$\{\psi, \chi\}$	0
$\{\chi, \chi\}^L$	0	$\{\chi, \chi\}^R$	0	$\{\chi, \chi\}$	0
$(\mathcal{G}, \mathcal{G})$	$(\mathcal{G}, B \oplus A \oplus A_{1,1 } . i)$	$(\mathcal{G}, B \oplus A \oplus A_{1,1 } . ii)$	$(\mathcal{G}, C_{p=1}^2 \oplus A_{1,1 } . i)$	$(\mathcal{G}, C_p^2 \oplus A_{1,1 } . i)$	$(\mathcal{G}, C_{\frac{1}{p}}^2 \oplus A_{1,1 } . ii)$
$\{x, y\}$	0	0	0	0	0
$\{x, \psi\}$	0	0	0	0	0
$\{x, \chi\}$	0	0	0	0	0
$\{y, \psi\}$	0	$-\psi$	$-\psi$	$-p\psi$	$-\frac{1}{p}\psi$
$\{y, \chi\}$	$-\chi$	0	$-\chi$	$-\chi$	$-\chi$
$\{\psi, \psi\}$	0	0	0	0	0
$\{\psi, \chi\}$	0	0	0	0	0
$\{\chi, \chi\}$	0	0	0	0	0

## 6 Manin supertriples and Drinfeld superdoubles

From the Manin supertriples of table IV we see that  $(gl(1|1), (C^3 + A)_{\epsilon_1}.i)$  and  $(gl(1|1), (C^3 + A)_{\epsilon_2}.ii)$  as Lie superalgebras are isomorphic, and so one can find an isomorphism that preserves also the bilinear form  $\langle ., . \rangle$ , so that they belong to the same Drinfeld superdouble. The isomorphism of Manin supertriples between  $(gl(1|1), (C^3 + A)_{\epsilon_1}.i)$  and  $(gl(1|1), (C^3 + A)_{\epsilon_2}.ii)$  is given by the following transformations

$$\begin{aligned} T'_1 &= -T_1 + cT_2 + dT_3, & T'_5 &= -aT_6, \\ T'_2 &= abT_2, & T'_6 &= -bT_5, \\ T'_3 &= -\frac{\epsilon_2 a^2}{\epsilon_1} T_3, & T'_7 &= -\frac{\epsilon_2 a}{\epsilon_1} T_8, \\ T'_4 &= eT_2 + fT_3 + \frac{\epsilon_2 a}{\epsilon_1 b} T_4, & T'_8 &= -\frac{\epsilon_2 a^2}{\epsilon_1 b} T_7, \end{aligned}$$

where  $(T_1, \dots, T_8)$  are generators of the Manin supertriple  $(gl(1|1), (C^3 + A)_{\epsilon_1}.i)$  and  $(T'_1, \dots, T'_8)$  are generators of the Manin supertriples  $(gl(1|1), (C^3 + A)_{\epsilon_2}.ii)$ . In same way we get the isomorphism matrices between all the Manin supertriples generated by  $gl(1|1)$  (see, appendix A). Furthermore, we deduce the following theorem:

**Theorem 7** *Drinfeld superdoubles generated by the  $gl(1|1)$  Lie superbialgebras of the type  $(4, 4)$  belong to the one of the following 6 classes and allows decomposition into all Lie superbialgebras listed in the class and their duals.*

$$\begin{aligned} \mathcal{Dsd}_{(4,4)}^1 &: \left( gl(1|1), I_{(2,2)} \right), \left( gl(1|1), C_{p=-1}^2 \oplus A_{1,1}.ii \right), \\ \mathcal{Dsd}_{(4,4)}^2 &: \left( gl(1|1), B \oplus A \oplus A_{1,1}.i \right), \left( gl(1|1), B \oplus A \oplus A_{1,1}.ii \right), \left( gl(1|1), C_{p=1}^2 \oplus A_{1,1}.i \right), \\ &\quad \left( gl(1|1), C_p^2 \oplus A_{1,1}.i \right), \left( gl(1|1), C_{\frac{1}{p}}^2 \oplus A_{1,1}.ii \right), \\ \mathcal{Dsd}_{(4,4)}^{3 \epsilon_1, \epsilon_2} &: \left( gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_1}.i \right), \left( gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_1}.ii \right), \quad \epsilon_1, \epsilon_2 = \pm 1, \\ \mathcal{Dsd}_{(4,4)}^{4 \epsilon_1, \epsilon_2} &: \left( gl(1|1), (C^3 + A)_{\epsilon_1}.i \right), \left( gl(1|1), (C^3 + A)_{\epsilon_2}.ii \right), \quad \epsilon_1, \epsilon_2 = \pm 1, \\ \mathcal{Dsd}_{(4,4)}^5 &: \left( gl(1|1), (C_{-1}^2 + A).i \right), \\ \mathcal{Dsd}_{(4,4)}^6 &: \left( gl(1|1), (C_0^5 + A).i \right). \end{aligned}$$

Drinfeld superdoubles generated by the  $gl(1|1)$  Lie superbialgebras  $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1$ , are spanned by the generators  $(T_1, \dots, T_8)$ , where  $\{T_1, T_2, T_3, T_4\}$  span the subspace  $\mathcal{D}_0$  of grade 0, and  $\{T_5, T_6, T_7, T_8\}$  span the subspace  $\mathcal{D}_1$  of grade 1. We give the generators (anti) commutation relations in table VII. Note that no one of these Drinfeld superdoubles are isomorphic to the  $osp(2|2) \cong sl(2|1)$  Lie superalgebra.

TABLE VII. Drinfeld superdoubles generated by the Lie superalgebra  $\mathcal{G} = gl(1|1)$ .

$(\mathcal{G}, \tilde{\mathcal{G}})$	Non-zero (anti) commutation relations
$(\mathcal{G}, I_{(2,2)})$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_5] = T_8, [T_4, T_6] = T_7,$ $\{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_8\} = iT_3$
$(\mathcal{G}, B \oplus A \oplus A_{1,1 } \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_7] = T_7, [T_4, T_6] = T_7,$ $[T_5, T_4] = T_5 - T_8, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = i(T_2 - T_3), \{T_6, T_8\} = iT_3$
$(\mathcal{G}, B \oplus A \oplus A_{1,1 } \cdot ii)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_6] = T_7 - T_6, [T_4, T_8] = T_8,$ $[T_5, T_4] = -T_8, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_8\} = i(T_2 + T_3)$
$(\mathcal{G}, C_{p=1}^2 \oplus A_{1,1 } \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_8] = T_8, [T_4, T_6] = T_7 - T_6,$ $[T_4, T_7] = T_7, [T_5, T_4] = T_5 - T_8, \{T_5, T_6\} = iT_2, \{T_6, T_8\} = i(T_2 + T_3), \{T_5, T_7\} = i(T_2 - T_3)$
$(\mathcal{G}, C_{p=-1}^2 \oplus A_{1,1 } \cdot ii)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_8] = -T_8, [T_4, T_6] = T_6 + T_7,$ $[T_4, T_7] = T_7, [T_5, T_4] = T_5 - T_8, \{T_5, T_6\} = iT_2, \{T_6, T_8\} = -i(T_2 - T_3), \{T_5, T_7\} = i(T_2 - T_3)$
$(\mathcal{G}, C_p^2 \oplus A_{1,1 } \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_8] = pT_8, [T_4, T_6] = T_7 - pT_6,$ $[T_4, T_7] = T_7, [T_5, T_4] = T_5 - T_8, \{T_5, T_6\} = iT_2, \{T_6, T_8\} = i(pT_2 + T_3), \{T_5, T_7\} = i(T_2 - T_3)$
$(\mathcal{G}, C_{\frac{1}{p}}^2 \oplus A_{1,1 } \cdot ii)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8, [T_4, T_8] = \frac{1}{p}T_8, [T_4, T_6] = T_7 - \frac{1}{p}T_6,$ $[T_4, T_7] = T_7, [T_5, T_4] = T_5 - T_8, \{T_5, T_6\} = iT_2, \{T_6, T_8\} = i(\frac{1}{p}T_2 + T_3), \{T_5, T_7\} = i(T_2 - T_3)$
$(\mathcal{G}, (B \oplus (A_{1,1} + A))_{\epsilon} \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -\epsilon T_5 - T_7, [T_1, T_8] = T_8, [T_4, T_8] = \epsilon T_8, [T_4, T_7] = \frac{\epsilon}{2}T_8,$ $[T_4, T_5] = T_8, [T_6, T_4] = \epsilon T_6 + \frac{\epsilon}{2}T_5 - T_7, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_7\} = i\frac{\epsilon}{2}T_2,$ $\{T_6, T_8\} = i(\epsilon T_2 + T_3), \{T_7, T_7\} = i\epsilon T_3$
$(\mathcal{G}, (B \oplus (A_{1,1} + A))_{\epsilon} \cdot ii)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = -\epsilon T_6 + T_8, [T_4, T_8] = -\frac{\epsilon}{2}T_7, [T_4, T_7] = \epsilon T_7,$ $[T_4, T_6] = T_7, [T_5, T_4] = \epsilon T_5 - \frac{\epsilon}{2}T_6 - T_8, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = i(\epsilon T_2 - T_3), \{T_5, T_8\} = -i\frac{\epsilon}{2}T_2,$ $\{T_6, T_8\} = iT_3, \{T_8, T_8\} = i\epsilon T_3$
$(\mathcal{G}, (C^3 + A)_{\epsilon} \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7, [T_1, T_8] = T_8 + \epsilon T_6, [T_4, T_8] = \frac{\epsilon}{2}T_7, [T_6, T_4] = -T_7,$ $[T_5, T_4] = \frac{\epsilon}{2}T_6 - T_8, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_5, T_8\} = i\frac{\epsilon}{2}T_2, \{T_6, T_8\} = iT_3,$ $\{T_8, T_8\} = -i\epsilon T_3$
$(\mathcal{G}, (C^3 + A)_{\epsilon} \cdot ii)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_7 - \epsilon T_5, [T_1, T_8] = T_8, [T_4, T_7] = \frac{\epsilon}{2}T_8, [T_6, T_4] = \frac{\epsilon}{2}T_5 - T_7,$ $[T_5, T_4] = -T_8, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_8\} = iT_3, \{T_7, T_7\} = i\epsilon T_3, \{T_6, T_7\} = i\frac{\epsilon}{2}T_2$
$(\mathcal{G}, (C_{-1}^2 + A) \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_5 - T_7, [T_1, T_8] = T_6 + T_8, [T_4, T_7] = \frac{1}{2}T_8, [T_4, T_8] = \frac{1}{2}T_7,$ $[T_5, T_4] = \frac{1}{2}T_6 - T_8, [T_6, T_4] = \frac{1}{2}T_5 - T_7, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_8\} = iT_3,$ $\{T_7, T_7\} = iT_3, \{T_6, T_7\} = \frac{1}{2}T_2, \{T_8, T_8\} = -iT_3, \{T_5, T_8\} = \frac{1}{2}T_2$
$(\mathcal{G}, (C_0^5 + A) \cdot i)$	$[T_1, T_5] = T_5, [T_1, T_6] = -T_6, [T_1, T_7] = -T_5 - T_7, [T_1, T_8] = T_8 - T_6, [T_4, T_7] = \frac{1}{2}T_8, [T_4, T_8] = -\frac{1}{2}T_7,$ $[T_5, T_4] = -T_8 - \frac{1}{2}T_6, [T_6, T_4] = \frac{1}{2}T_5 - T_7, \{T_5, T_6\} = iT_2, \{T_5, T_7\} = -iT_3, \{T_6, T_8\} = iT_3,$ $\{T_7, T_7\} = iT_3, \{T_6, T_7\} = \frac{1}{2}T_2, \{T_8, T_8\} = iT_3, \{T_5, T_8\} = -\frac{1}{2}T_2$

## 7 Integrable system on the $(2|2)$ -dimensional homogeneous superspace $OSp(1|2)/U(1)$

In this section, we briefly discuss about the relationship between the GCYBE and the theory of classical integrable systems and construct a certain dynamical system on the supersymplectic supermanifold  $OSp(1|2)/U(1)$ . We begin by recalling the notion of a completely integrable Hamiltonian system. A Hamiltonian system is modelled by a Poisson supermanifold  $\mathcal{M}$  ( $\dim \mathcal{M}$  is denote by  $(d_B|d_F)$  where  $d_B(d_F)$  is the dimension of the bosonic (fermionic) part.) and a Hamiltonian  $\mathcal{H} \in C^\infty(\mathcal{M})$ , such that its time-evolution is given by

$$X_{\mathcal{H}} = \{\mathcal{H}, f\}, \quad (35)$$

where  $X_{\mathcal{H}}$  is the Hamiltonian supervector field on  $\mathcal{M}$  corresponding to  $\mathcal{H}$  and  $f \in C^\infty(\mathcal{M})$ . Whenever  $m(t)$  be any integral curve of  $X_{\mathcal{H}}$ , then  $(df/dt)(m(t)) = \{\mathcal{H}, f\}(m(t))$ . In particular, an observable  $f$  is conserved, or a constant of the motion, if  $\{\mathcal{H}, f\} = 0$ , or more generally, two observables  $f_1$  and  $f_2$  on a Poisson supermanifold  $\mathcal{M}$  are in involution if  $\{f_1, f_2\} = 0$ . Note that every integral curve  $m(t)$  of  $X_{\mathcal{H}}$  lies entirely in some supersymplectic leaf of  $\mathcal{M}$ . Hence, we might as well assume that  $\mathcal{M}$  is a supersymplectic supermanifold.

**Definition 8** *The dynamical system defined on a  $(d_B|d_F)$ -dimensional supersymplectic supermanifold  $\mathcal{M}$  (here  $d_B$  and  $d_F$  are even numbers) by a Hamiltonian  $\mathcal{H} \in C^\infty(\mathcal{M})$  is completely integrable if there exist  $n = \frac{1}{2}\dim\mathcal{M}$  independent conserved quantities  $f_1, \dots, f_n \in C^\infty(\mathcal{M})$  in involution.*

We now give a general procedure for constructing completely integrable Hamiltonian systems starting from a classical r-matrix. Let  $\mathcal{M}$  be a supermanifold with a non-degenerate supersymplectic form  $\omega$ , and  $x^\mu$  ( $\mu = 1, \dots, \dim\mathcal{M}$ ) be the locally coordinates of  $\mathcal{M}$ . Consider Poisson superbracket structure on  $\mathcal{M}$  for arbitrary functions  $f(x^\mu)$  and  $g(x^\nu)$  as [21]

$$\{f, g\} = \frac{f \overleftarrow{\partial}}{\partial x^\mu} \omega^{\mu\nu} \frac{\overrightarrow{\partial} g}{\partial x^\nu} = (-1)^{\nu(\mu+|f|)} \omega^{\mu\nu} \frac{\overrightarrow{\partial} f}{\partial x^\mu} \frac{\overrightarrow{\partial} g}{\partial x^\nu}, \quad (36)$$

where  $\omega^{\mu\nu}$  is the superinverse of the  $\omega_{\mu\nu}$ , such that  $\omega^{\mu\nu} = -(-1)^{\mu\nu} \omega^{\nu\mu}$ . Then, by introducing dynamical variables  $S_i(x^\mu)$ ,  $i = 1, \dots, \dim\mathcal{G}$ , as

$$\{S_i, S_j\} = f_{ij}^k S_k, \quad (37)$$

and the defining the Lie superalgebra-valued function

$$Q(x) = (-1)^j S_i(x) r^{ij} X_j, \quad (38)$$

and using the fact that  $r$  satisfies the GCYBE, it is concluded that [21]

$$\{Str[(Q(x))^n], Str[(Q(xy))^m]\} = 0, \quad 0 < m, n \in \mathbb{Z}^+. \quad (39)$$

Therefore, the coefficients

$$I_k(x) = Str[(Q(x))^k], \quad 0 < k \in \mathbb{Z}^+, \quad (40)$$

are regarded as constants of motion of a certain dynamical system. In the following by using the above statements we construct an integrable system on a supersymplectic homogeneous superspace. It has been shown that  $OSp(1|2)$  coherent states are parametrized by points of a supersymplectic supermanifold, namely the homogeneous superspace  $OSp(1|2)/U(1)$ , such that  $OSp(1|2)/U(1)$  is a supercoadjoint orbit of  $OSp(1|2)$ , i.e., a superunit disc [22]. This superunit disc can be equipped with a supersymplectic even two-form  $\omega$  that is given by [22]

$$\omega = -2i\tau \left[ 1 + \frac{1}{2}\theta^\# \theta \frac{(1+|z|^2)}{(1-|z|^2)} \right] \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} + i\tau \frac{d\theta \wedge d\theta^\#}{(1-|z|^2)} + i\tau\theta\bar{z} \frac{dz \wedge d\theta^\#}{(1-|z|^2)^2} - i\tau z\theta^\# \frac{d\theta \wedge d\bar{z}}{(1-|z|^2)^2}. \quad (41)$$

where  $\tau$  corresponds to each irreducible representation of  $OSp(1|2)$ . Here  $z$  is a complex number and  $\theta$  is an odd coordinate, i.e. an anticommuting Grassmann number. Note that  $\bar{z}$  is the usual complex conjugate of  $z$  while  $\theta^\#$  is the so-called adjoint of  $\theta$  [one can write them in terms of real coordinates  $x, y$  and  $\psi, \chi$  as  $z = x + iy$ ,  $\bar{z} = x - iy$  and  $\theta = \psi + i\chi$ ,  $\theta^\# = -i(\psi - i\chi)$  [22].].

Now, using the definition of two-form [18] as  $\omega = \frac{(-1)^{\mu\nu}}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$ , one can find the superinverse of the  $\omega_{\mu\nu}$  as follows:

$$\omega^{\mu\nu} = \frac{1}{4i\tau} \begin{pmatrix} 0 & \frac{(2(1-|z|^2)-\theta^\#\theta)(1-|z|^2)}{2} & 0 & -z\theta^\#(1-|z|^2) \\ -\frac{(2(1-|z|^2)-\theta^\#\theta)(1-|z|^2)}{2} & 0 & \bar{z}\theta(1-|z|^2) & 0 \\ 0 & -\bar{z}\theta(1-|z|^2) & 0 & 2(1-|z|^2)-z\bar{z}\theta^\#\theta \\ z\theta^\#(1-|z|^2) & 0 & 2(1-|z|^2)-z\bar{z}\theta^\#\theta & 0 \end{pmatrix}. \quad (42)$$

Thus, by taking the structure constants of the  $gl(1|1)$  (20) and employing the relations (36) and (37) we then obtain the following system of PDEs

$$\frac{\partial S_1}{\partial \theta} \frac{\partial S_2}{\partial \theta^\#} + \frac{\partial S_1}{\partial \theta^\#} \frac{\partial S_2}{\partial \theta} - \frac{(1-|z|^2)}{2} \left( \frac{\partial S_1}{\partial z} \frac{\partial S_2}{\partial \bar{z}} - \frac{\partial S_1}{\partial \bar{z}} \frac{\partial S_2}{\partial z} \right) = 0, \quad (43)$$

$$\frac{\partial S_2}{\partial \theta} \frac{\partial S_3}{\partial \theta^\#} + \frac{\partial S_2}{\partial \theta^\#} \frac{\partial S_3}{\partial \theta} - \frac{(1-|z|^2)}{2} \left( \frac{\partial S_2}{\partial z} \frac{\partial S_3}{\partial \bar{z}} - \frac{\partial S_2}{\partial \bar{z}} \frac{\partial S_3}{\partial z} \right) = 0, \quad (44)$$

$$\frac{\partial S_2}{\partial \theta} \frac{\partial S_4}{\partial \theta^\#} + \frac{\partial S_2}{\partial \theta^\#} \frac{\partial S_4}{\partial \theta} - \frac{(1-|z|^2)}{2} \left( \frac{\partial S_2}{\partial z} \frac{\partial S_4}{\partial \bar{z}} - \frac{\partial S_2}{\partial \bar{z}} \frac{\partial S_4}{\partial z} \right) = 0, \quad (45)$$

$$-2(1-|z|^2) \left( \frac{\partial S_1}{\partial \theta} \frac{\partial S_3}{\partial \theta^\#} + \frac{\partial S_1}{\partial \theta^\#} \frac{\partial S_3}{\partial \theta} \right) + (1-|z|^2)^2 \left( \frac{\partial S_1}{\partial z} \frac{\partial S_3}{\partial \bar{z}} - \frac{\partial S_1}{\partial \bar{z}} \frac{\partial S_3}{\partial z} \right) = 4i\tau S_3, \quad (46)$$

$$-2(1-|z|^2) \left( \frac{\partial S_1}{\partial \theta} \frac{\partial S_4}{\partial \theta^\#} + \frac{\partial S_1}{\partial \theta^\#} \frac{\partial S_4}{\partial \theta} \right) + (1-|z|^2)^2 \left( \frac{\partial S_1}{\partial z} \frac{\partial S_4}{\partial \bar{z}} - \frac{\partial S_1}{\partial \bar{z}} \frac{\partial S_4}{\partial z} \right) = -4i\tau S_4, \quad (47)$$

$$2(1-|z|^2) \left( \frac{\partial S_3}{\partial \theta} \frac{\partial S_4}{\partial \theta^\#} + \frac{\partial S_3}{\partial \theta^\#} \frac{\partial S_4}{\partial \theta} \right) + (1-|z|^2)^2 \left( \frac{\partial S_3}{\partial z} \frac{\partial S_4}{\partial \bar{z}} - \frac{\partial S_3}{\partial \bar{z}} \frac{\partial S_4}{\partial z} \right) = 4i\tau S_2, \quad (48)$$

$$\frac{\partial S_3}{\partial \theta} \frac{\partial S_3}{\partial \theta^\#} + \frac{(1-|z|^2)}{2} \frac{\partial S_3}{\partial z} \frac{\partial S_3}{\partial \bar{z}} = 0, \quad (49)$$

$$\frac{\partial S_4}{\partial \theta} \frac{\partial S_4}{\partial \theta^\#} + \frac{(1-|z|^2)}{2} \frac{\partial S_4}{\partial z} \frac{\partial S_4}{\partial \bar{z}} = 0. \quad (50)$$

Here  $\frac{\partial}{\partial x^\mu}$  stands for left derivative with respect to  $x^\mu$ . A special class of the solutions for the above equations is given by

$$S_1 = \frac{2i\tau\theta^\#\theta}{(1-|z|^2)}, \quad S_2 = \frac{\alpha_0\beta_0}{2i\tau}(1-|z|^2), \quad S_3 = \alpha_0\theta^\#, \quad S_4 = \beta_0\theta, \quad (51)$$

where  $\alpha_0$  and  $\beta_0$  are even real and pure imaginary constants, respectively, so  $\alpha_0\beta_0 = i\mathcal{C}$  and  $\mathcal{C}$  is a real constant. On the other hand, among the coboundary Lie superbialgebras listed in table V, only the  $(gl(1|1), C_{p=-1}^2 \oplus A_{1,1}|.ii)$  satisfies the GCYBE, therefore, using the classical  $r$ -matrix concerning this Lie superbialgebra (table V), the relation (38) takes on the following form

$$Q = S_1X_2 - S_2X_1. \quad (52)$$

It is easy to verify that the following supermatrices provide a  $(2|2)$ -dimensional representation of  $gl(1|1)$

$$\begin{aligned} X_1 &= \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n-1 & 0 \\ 1 & 0 & -1 & n+1 \end{pmatrix}, & X_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ie & ie & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ie & 0 & -\frac{ie}{2} & ie \end{pmatrix}, \\ X_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ie & 0 & \frac{ie}{2} & -ie \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (53)$$

where  $e$  and  $n$  are real constants. Thus, substituting the above relations into (52) and with the help of (40), we obtain the constants of motion as follows:

$$I_2(x^\mu) = -\mathcal{C} \left[ \frac{\mathcal{C}}{2\tau^2}(1-x^2-y^2)^2 + 4e\psi\chi \right], \quad (54)$$

$$I_3(x^\mu) = \frac{3\mathcal{C}^2}{\tau} \left[ \frac{n\mathcal{C}}{4\tau^2}(1-x^2-y^2)^3 + e(2n+1)\psi\chi(1-x^2-y^2) \right], \quad (55)$$

where  $x, y, \psi$  and  $\chi$  are the superphase space coordinates. One can consider  $I_2(x^\mu)$  as the Hamiltonian of the system and  $I_3(x^\mu)$  as other constant of motion. Note that  $I_1(x^\mu)$  is also vanished.

## 8 Conclusion

In this paper, we first classified decomposable Lie superalgebras of the type  $(2, 2)$ . Then, we obtained all  $gl(1|1)$  Lie superbialgebras. In this respect, we determined their types (triangular, quasi-triangular or factorizable) and also classified Drinfeld superdoubles generated by the  $gl(1|1)$  Lie superalgebra as a theorem. Using this classification one can investigate super Poisson-Lie symmetry of the sigma models (specially in the WZW model) on the  $\mathbf{GL}(1|1)$  Lie supergroup. Furthermore, one can construct string cosmology models with super Poisson-Lie symmetry [23] on the  $\mathbf{GL}(1|1)$  Lie supergroup. By applying the procedure of constructing integrable systems [21], we found a new integrable system on the supersymplectic homogeneous superspace  $OSp(1|2)/U(1)$ . Here, we have used of the coboundary Lie superbialgebra of the type triangular, i.e.,  $(gl(1|1), C_{p=-1}^2 \oplus A_{1,1|}.ii)$ . In the same way, one can construct other integrable systems on the  $OSp(1|2)/U(1)$  superspace using the classical  $r$ -matrices concerning the  $(3|2)$  and  $(4|4)$ -dimensional coboundary Lie superbialgebras [11, 12]. Some of these remarks are under investigation.

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## Appendix A : Isomorphism matrices between the Manin supertriples based on the $gl(1|1)$

- $\mathcal{Dsd}_{(4,4)}^1 :$   $(gl(1|1), I_{(2,2)}) \longrightarrow (gl(1|1), C_{p=-1}^2 \oplus A_{1,1|}.ii)$

$$C = \begin{pmatrix} 1 & m & n & 0 & 0 & 0 & 0 & 0 \\ 0 & bc & cd - be & 0 & 0 & 0 & 0 & 0 \\ 0 & bc & abc - be + cd & 0 & 0 & 0 & 0 & 0 \\ -1 & r & s & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & c & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ab \end{pmatrix}, \quad a, b, c \in \mathfrak{R} - \{0\}; \quad e, d, m, n, r, s \in \mathfrak{R},$$

- $\mathcal{Dsd}_{(4,4)}^{2p} :$

$$(gl(1|1), B \oplus A \oplus A_{1,1|}.i) \longrightarrow (gl(1|1), B \oplus A \oplus A_{1,1|}.ii)$$

$$C = \begin{pmatrix} 1 & m & n & 2 & 0 & 0 & 0 & 0 \\ 0 & ad & -ad + bc & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -bc & 0 & 0 & 0 & 0 & 0 \\ 0 & r & s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & d & c & -c & -d \\ 0 & 0 & 0 & 0 & 0 & c & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \end{pmatrix}, \quad a, b, c, d \in \mathfrak{R} - \{0\}; \quad m, n, r, s \in \mathfrak{R},$$



$$(gl(1|1), B \oplus A \oplus A_{1,1|1}.i) \longrightarrow (gl(1|1), C_{p=1}^2 \oplus A_{1,1|1}.i)$$

$$C = \begin{pmatrix} 1 & m & n & 0 & 0 & 0 & 0 & 0 \\ 0 & a(b+d) & bc-ad & 0 & 0 & 0 & 0 & 0 \\ 0 & a(b+d) & -2ab-bc-ad & 0 & 0 & 0 & 0 & 0 \\ -1 & r & s & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & b & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 2b & -2b & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 0 & -2a \end{pmatrix}, \quad \begin{cases} a+c \neq 0, \\ b+d \neq 0, \end{cases} \quad \begin{cases} a, b \in \mathfrak{R} - \{0\}, \\ m, n, r, s, c \in \mathfrak{R}, \end{cases}$$

$$(gl(1|1), B \oplus A \oplus A_{1,1|1}.i) \longrightarrow (gl(1|1), C_p^2 \oplus A_{1,1|1}.i)$$

$$C = \begin{pmatrix} 1 & m & n & 2 & 0 & 0 & 0 & 0 \\ 0 & ad & -ad+bc & 0 & 0 & 0 & 0 & 0 \\ 0 & ad & -pbc-ad & 0 & 0 & 0 & 0 & 0 \\ -1 & r & s & p-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c \\ 0 & 0 & 0 & 0 & a & b & -b & -a \\ 0 & 0 & 0 & 0 & 0 & (1+p)b & -(1+p)b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1+p)d & 0 \end{pmatrix}, \quad a, b, c, d \in \mathfrak{R} - \{0\}; \quad m, n, r, s \in \mathfrak{R},$$

$$(gl(1|1), B \oplus A \oplus A_{1,1|1}.i) \longrightarrow (gl(1|1), C_{\frac{1}{p}}^2 \oplus A_{1,1|1}.ii)$$

$$C = \begin{pmatrix} 1 & m & n & 2 & 0 & 0 & 0 & 0 \\ 0 & ac & -ac-bd & 0 & 0 & 0 & 0 & 0 \\ 0 & ac & \frac{1}{p}bd-ac & 0 & 0 & 0 & 0 & 0 \\ -1 & r & s & \frac{1}{p}-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & -d & d & -c \\ 0 & 0 & 0 & 0 & 0 & \frac{-(1+p)d}{p} & \frac{(1+p)d}{p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(1+p)a}{p} & 0 \end{pmatrix}, \quad a, b, c, d \in \mathfrak{R} - \{0\}; \quad m, n, r, s \in \mathfrak{R},$$

$$\bullet \mathcal{D}sd_{(4,4)}^3{}^{\epsilon_1, \epsilon_2} :$$

$$(gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_1=1}.i) \longrightarrow (gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_2=-1}.i)$$

$$C = \begin{pmatrix} 1 & m & n & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix}, \quad a \in \mathfrak{R} - \{0\}; \quad m, n, b, c \in \mathfrak{R},$$

$$\left( gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_1} . i \right) \longrightarrow \left( gl(1|1), (B \oplus (A_{1,1} + A))_{\epsilon_2} . ii \right)$$

$$C = \begin{pmatrix} -1 & m & n & 0 & 0 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_2}{\epsilon_1} a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & \frac{\epsilon_2}{\epsilon_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_2}{\epsilon_1} a \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\epsilon_2}{\epsilon_1} a & 0 \end{pmatrix}, \quad a \in \mathbb{R} - \{0\}; \quad m, n, b, c \in \mathbb{R},$$

•  $\mathcal{D}sd_{(4,4)}^{4\epsilon_1, \epsilon_2} :$

$$\left( gl(1|1), (C^3 + A)_{\epsilon_1=1} . i \right) \longrightarrow \left( gl(1|1), (C^3 + A)_{\epsilon_2=-1} . i \right)$$

$$C = \begin{pmatrix} 1 & m & n & 0 & 0 & 0 & 0 & 0 \\ 0 & -ab^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2b^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & d & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ab & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a^2b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ab \end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad m, n, c, d \in \mathbb{R}.$$

$$\left( gl(1|1), (C^3 + A)_{\epsilon_1} . i \right) \longrightarrow \left( gl(1|1), (C^3 + A)_{\epsilon_2} . ii \right)$$

$$C = \begin{pmatrix} -1 & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & ab & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon_2 a^2}{\epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & \frac{\epsilon_2 a}{\epsilon_1 b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_2 a}{\epsilon_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_2 a^2}{\epsilon_1 b} & 0 \end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, f \in \mathbb{R}.$$

## References

- [1] V. G. Drinfel'd, *Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Sov. Math. Dokl. **27** (1983), 68-71.
- [2] P. Delorme, *Sur les triples de Manin pour les complexes*, Jour. Alg. **246** (2001) 97-174.
- [3] M.A. Jafarizadeh and A. Rezaei-Aghdam, *Poisson-Lie T-duality and Bianchi type algebras*, Phys. Lett. **B458** (1999) 477-490, [[arXiv:hep-th/9903152](#)].
- [4] A. Rezaei-Aghdam, M. Hemmati, A. R. Rastkar, *Classification of real three-dimensional Lie bialgebras and their Poisson-Lie groups*, J. Phys. **A: Math.Gen.** **38** (2005) 3981-3994, [[arXiv:math-ph/0412092](#)].
- [5] X. Gomez, *Classification of three-dimensional Lie bialgebras*, J. Math. Phys. **41** (2000) 493956.

- [6] Y. Kosmann-Schwarzbach, *Lie bialgebras, Poisson-Lie groups and dressing transformations integrability of nonlinear systems*: Proc.(Pondicherry) edited by Y. Kosmann-Schwarzbach, B. Grammaticos and K. M. Tamizhmani (Springer, Berlin, 1996), pp. 104-70.
- [7] N. Andruskiewitsch, *Lie superbialgebras and Poisson-Lie supergroups*, Abh. Math. Sem. Univ. Hamburg **63** (1993) 147-163.
- [8] N. Beisert and E. Spill, *The classical  $r$ -matrices of AdS/CFT and its Lie bialgebras structure*, Commun. Math. Phys. **285** (2009) 537-565, [arXiv:0708.1762 [hep-th]].
- [9] A. Eghbali, A. Rezaei-Aghdam, *Poisson-Lie  $T$ -dual sigma models on supermanifolds*, JHEP **09** (2009) 094, [arXiv:0901.1592 [hep-th]].
- [10] N. Geer, *Etingof-Kazdan quantization of Lie super bialgebras*, To appear in Advances of Mathematics, [arXiv:math.QA/0409563].
- [11] C. Juszczak and J. T. Sobczyk, *Classification of low dimensional Lie super-bialgebras*, J. Math. Phys. **39**, (1998), 4982-4992.
- [12] C. Juszczak, *Classical  $r$ -matrices for the  $osp(2|2)$  Lie superalgebra*, J. Math. Phys. **41**, 2350 (2000).
- [13] A. Eghbali, A. Rezaei-Aghdam and F. Heidarpour, *Classification of two and three dimensional Lie superbialgebras*, J. Math. Phys. **51**, 073503 (2010), [arXiv:0901.4471 [math-ph]].
- [14] L. Frappat, V. Hussin and G. Rideau, *Classification of the quantum deformations of the superalgebra  $gl(1|1)$* , J. Phys. A: Math.Gen. **31** (1998) 4049-4072, [arXiv:math-ph/0412092].
- [15] V. Shomerus and H. Saleur, *The  $GL(1|1)$  WZW model: from supergeometry to logarithmic CFT*, Nucl. Phys. **B734** (2006), 221-245, [arXiv:hep-th/0510032].
- [16] T. Creutzig, *Branes in Supergroups* (Ph.D. thesis), [arXiv:0908.1816 [hep-th]].
- [17] N. Backhouse, *A classification of four-dimensional Lie superalgebras*, J. Math. Phys. **19** (1978), 2400-2402.
- [18] B. DeWitt, *Supermanifolds*, Cambridge University Press 1992.
- [19] A. Eghbali, A. Rezaei-Aghdam and F. Heidarpour, *Classification of four and six dimensional Drinfel'd superdoubles*, J. Math. Phys. **51**, 103503 (2010), [arXiv:0911.1760 [math-ph]].
- [20] A. Eghbali and A. Rezaei-Aghdam, *Classical  $r$ -matrices of two and three dimensional Lie super-bialgebras and their Poisson-Lie supergroups*, accepted for publication in Theor. Math. Phys., [arXiv:0908.2182 [math-ph]].
- [21] R. B. Zhang, M. D. Gould and A. J. Bracken, *Solutions of the graded classical Yang-Baxter equation and integrable models*, J. Phys. A: Math.Gen. **24** (1991) 1185-1197.
- [22] Amine M. El Gradechi, *On the supersymplectic homogeneous superspace underlying the  $OSp(1/2)$  coherent states*, J. Math. Phys. **34** (1993) 5951-5963, [arXiv:hep-th/9301132].
- [23] A. Eghbali and A. Rezaei-Aghdam, *String cosmology from Poisson-Lie  $T$ -dual sigma models on supermanifolds*, JHEP **01** (2012) 151, [arXiv:1107.2041 [hep-th]].